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NEW PROPERTIES OF ALL REAL FUNCTIONS

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In a former paper[1] the author communicated a number of properties of every real function $f(x)$, which were stated in terms of the successive saltus functions associated with a given function. The present results do not involve these saltus functions and are direct qualifications of $f(x)$. Since $f(x)$ is entirely unrestricted, except, of course, that it is defined—even this restriction may be partially dispensed with—and therefore finite for every real x, these qualifications are consequences of nothing else than that $f(x)$ is a function. A new light is thus thrown upon the nature of a function.

The new properties are of two kinds, descriptive and metric; the former are concerned with density and the latter with measure (Lebesgue).

For the sake of greater concreteness, we discuss, for the most part, planar sets and real, single-valued functions of two real variables.

1. *Descriptive Properties.*—We say that a planar set S is an "I-region" (= open set) if every point of S is an inner point of S. We deal with binary relations $\mathfrak{R}$ between I-regions and points. $I\mathfrak{R}P$ shall mean that the I-region I has the relation $\mathfrak{R}$ to the point P. The relation $\mathfrak{R}$ is said to be "closed" if the relationships $I\,\mathfrak{R}\,P_n$ and $\lim_{n\to\infty} P_n = P$ imply $I\,\mathfrak{R}\,P$. By a "neighborhood" of a point P, we understand an I-region containing P; by a "partial neighborhood" of P, an I-region of which P is an inner or a boundary point.

We have the following

Lemma I. If $\mathfrak{R}$ is a closed relation, then the points for which (a) $N\mathfrak{R}P$ for every neighborhood N of P, and (b) a partial neighborhood N_ϵ exists such that $\overline{N_\epsilon\,\mathfrak{R}\,P}$ (i.e., $N_\epsilon\,\mathfrak{R}\,P$ is false) *constitute a non-dense* (i.e., nowhere dense) *set.*

An important example of a closed relation appears in connection with

a function. Let $z = f(x, y)$ be any given one-valued function of the two real variables x and y, and let f be defined for the entire XY plane. Let $\mathfrak{R} = \mathfrak{R}_{r_1 r_2}$, where $r_1 < r_2$ are two real numbers, be defined as follows: $I\mathfrak{R}_{r_1 r_2} P$ if and only if an infinite sequence of points P_n of the I-region I exists such that $\lim_{n \to \infty} P_n = P$, $\lim_{n \to \infty} f(P_n)$ exists, and $r_1 \leqq \lim_{n \to \infty} f(P_n) \leqq r_2$; here $P, P_n = (x, y), (x_n, y_n)$ and $f(P), f(P_n) = f(x, y), f(x_n, y_n)$.

Definition. The function $f(x, y)$ is said to be "densely approached at the point (ξ, η)" or in other words, the point $(\xi, \eta, f(\xi, \eta))$ of the "surface" $z = f(x, y)$ is said to be densely approached, if for every positive ϵ there exists a planar neighborhood N of (ξ, η) such that the points of N for which $|f(x, y) - f(\xi, \eta)| < \epsilon$ form a dense set in N.

Definition. An "exhaustible" set (= set of first category, according to Baire) is the sum of $\aleph_0$ non-dense sets; a "residual" set, the complement (with respect to the XY plane) of an exhaustible set.[2]

By the aid of Lemma I and the closed relation $\mathfrak{R}_{r_1 r_2}$, we prove

Theorem I. For every real function $f(x, y)$ whatsoever, the points of the surface $z = f(x, y)$ that are densely approached form a residual set. Conversely, given any residual set R whatsoever, a function $f(x, y)$ exists that is densely approached at and only at the points of R.

The following definition of dense approach is equivalent: The function $f(x, y)$ is said to be densely approached at $P = (\xi, \eta)$ if, for every partial neighborhood $N_<$ of P, the set of points of the surface $z = f(x, y)$ corresponding to the points of $N_<$ has $(\xi, \eta, f(\xi, \eta))$ as a limit point. By the use of this definition, we get the following theorem, which is equivalent to Theorem I, but which shows better, perhaps, the remarkable degree of "microscopic symmetry" an unconditioned function possesses.

Theorem I'. With every function $f(x, y)$ whatsoever, there is associated a residual set R—dependent on f—of the XY plane such that if $P = (\xi, \eta)$ is a point of R and $N_<$, a partial neighborhood of P, then $(\xi, \eta, f(\xi, \eta))$ is a limit point of the set of points $(x, y, f(x, y))$ for which (x, y) is in $N_<$.

Definition. The function f is said to be "inexhaustibly approached" at the point P if every neighborhood of P contains, for every $\epsilon > 0$, an inexhaustible set of points—i.e., a set that is not exhaustible—at which f differs from $f(P)$ by less than ϵ.

If M is any planar set, we use, in connection with approach, the expression "via M" to designate that (x, y) is restricted to range in M. Thus "f is inexhaustibly approached at P via M" means that for every neighborhood N of P and every $\epsilon > 0$, the set MN, which is the aggregate of points common to M and N, contains an inexhaustible set of points at which f differs from $f(P)$ by less than ϵ.

We have the following

Theorem II. For every function $f(x, y)$, there exists in the XY plane a

residual set R—dependent on f—such that if P is a point of R, and N_ϵ, a partial neighborhood of P, the function f is inexhaustibly, and therefore densely approached at P via RN_ϵ.

With the aid of Theorem II we prove

Theorem III. With every function $f(x, y)$ there is associated—not uniquely, however—a dense set D of the XY plane, such that f is continuous if (x, y) ranges over D.

2. *Generalizations.*—The considerations and theorems of Section 1 apply not only to functions of n real variables, but to every space S that satisfies the following four conditions:

(1) S is metric;[3] that is to say, with every pair of elements P and Q of S there is associated a non-negative, real number $\overline{PQ}$ (Fréchet's écart) in such a way that if P, Q and R are three elements of S, then

(*a*) $\overline{PQ} = \overline{QP}$;

(*b*) $\overline{PQ} = 0$, when and only when $P = Q$; and

(*c*) $\overline{PQ} + \overline{QR} \geqq \overline{PQ}$.

(2) S is complete (vollständig);[4] that is to say, if $\{P_1, P_2, \ldots, P_n \ldots\}$ is a "regular" sequence of elements of S—in other words, for every $\epsilon > 0$ there exists an integer n_ϵ such that $\overline{P_\lambda P_\mu} < \epsilon$ for $\lambda > n_\epsilon$ and $\mu > n_\epsilon$—there exists a limit element P (i.e., an element P with the property $\lim_{n \to \infty} \overline{P_n P} = 0$).

(3) S contains a denumerable subset that is dense in S.

(4) S has no isolated points.

We may thus state the following theorem—the definitions of the terms for S may be obtained after slight and evident modifications from those for the plane.

Theorem IV. Let S be any complete, metric space containing a dense, denumerable subset and without isolated points; and f(P), any real function defined for the elements P of S. Then there exists a residual set R, such that if P is a point of R, and N_ϵ a partial neighborhood of P, the function f is inexhaustibly and therefore densely approached at P via RN_ϵ. Also there exists a dense subset D of S such that f (P) is continuous if P ranges over D.

As particular examples of a complete, metric space, with a dense, denumerable subset and without isolated points, we mention:

(*a*) Euclidean n-space where the écart between two points is the euclidean distance between them.

(*b*) A perfect subset of euclidean space.

(*c*) Hilbert space, that is, the ensemble of sequences $(x_1, x_2, \ldots x_n, \ldots)$ of real numbers with convergent $\sum_{n=1}^{n=\infty} x_n^2$. The écart between two "points" $(x_1, x_2, \ldots x_n, \ldots)$ and $(y_1, y_2, \ldots y_2, \ldots)$ is defined to be $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \ldots}$.

(*d*) Function space: S consists of all real, continuous functions $f(x)$

defined for $0 \leqq x \leqq 1$. The écart between $f_1(x)$ and $f_2(x)$ is defined to be $\max|f_1(x)-f_2(x)|$.

The assumption that f is single-valued may also be dropped without invalidating Theorems I and II—Theorem III, of course implies single-valuedness by its very nature. We thus get the following generalization of Theorem I (and a similar one for Theorem II).

Theorem I [a]. *Let $f(x, y)$ be any real function defined for the entire XY plane and taking at every point at least one value—the number of values may change, however, from point to point and vary from 1 to c, the cardinal number of the continuum. Then the points (x, y) such that every surface point $(x, y, f(x, y))$ is densely approached by the surface $z = f(x, y)$ constitute a residual set.*

3. *Metric Properties.*—As in the case of Section 1, we discuss in this section planar sets and functions of two variables.

Let S be any planar set; P, a point of S; C_r, a circle with P as center and r as radius; $m(C_r)$, the area of C_r, and $m_e(SC_r)$, the exterior Lebesgue measure of the portion of S in C_r. Then if

$$\lim_{r \to 0} \frac{m_e(SC_r)}{m(C_r)}$$

exists and is equal to k, we say that the "exterior metric density" of S at the point P is k. We have the following

Theorem V.[5] *Let S be any planar set. Then the points of S at which the exterior metric density of S is $\neq 1$*—i.e., the points where the exterior metric density either does not exist or does exist and is less than 1—*constitute a set of zero measure (Lebesgue).*

Definition. $N_<$ is said to be a "*non-vanishing* partial neighborhood of P," if the exterior metric density of $N_<$ at P is $\neq 0$. We have the following lemma, which corresponds to Lemma I for the descriptive properties:

Lemma II. Let $\mathfrak{R}$ be a closed relation as in Lemma I. The points P for which (a) $N \mathfrak{R} P$ for every neighborhood of P and (b) a non-vanishing partial neighborhood $N_<$ exists such that $\overline{N_< \mathfrak{R} P}$ (i.e., $N_< \mathfrak{R} P$ is false) constitute a set of zero measure.

By the aid of this lemma we prove

Theorem VI. Let $f(x, y)$ be any real, one-valued function defined in the entire plane. Then there exists in the XY plane a set Z—dependent on f—of measure zero, such that if (a) (x, y) is any point of the XY plane not belonging to Z; (b) $N_<$, any non-vanishing partial neighborhood of (x, y); and (c) S, any sphere with $(x, y, f(x, y))$ as center; then there is at least one point of the surface $z = f(x, y)$ lying in the sphere S and having as projection upon the XY plane a point in $N_<$.

This theorem becomes false if we omit the restriction that the partial neighborhood $N_<$ shall be non-vanishing.

By means of Lemma II, we prove also

Theorem VII. The set of points P at which f is inexhaustibly approached and for which a non-vanishing partial neighborhood exists via which f is exhaustibly approached constitute a set of measure zero.

Definition. f is said to be "neglectably approached at the point $P \equiv (\xi,\eta)$ via M" if a sphere S exists with $(\xi, \eta, f(\xi, \eta))$ as center, such that the projection upon the XY plane of the set of points common to S and the surface $z = f(x, y)$ has a set of measure zero in common with M.

The following theorem generalizes Theorem VI.

Theorem VIII. Let $z = f(x, y)$ be any real, one-valued function defined in the XY plane. Then the points P of the XY plane that possess a non-vanishing partial neighborhood via which f is neglectably approached at P constitute a set of zero measure.

Definition. f is "quasi-continuous"[6] at P if for every ϵ the set of points Q for which $|f(Q) - f(P)| < \epsilon$ has 1 as exterior metric density at P.

We have the following theorem, which generalizes theorem *VIII*.

Theorem IX. f is quasi-continuous except at the points of a set of measure zero.

Concluding Remarks.—As in the case of the descriptive properties, the metric theorems may be extended to many-valued functions. Theorem VIII, for example, when thus generalized reads as follows: *Let $z = f(x, y)$ be any real, single- or many-valued function defined in the entire XY plane. Then the points (x, y) of the XY plane for which a surface point $(x, y, f(x, y))$ and a non-vanishing partial neighborhood N_ϵ exist such that $(x, y, f(x, y))$ is neglectably approached via N_ϵ constitute a set of zero measure.*

The metric properties hold for functions of a single variable and, in general, for functions of n variables. Extension to $\aleph_0$ space, to function space and to more general spaces would require a satisfactory definition of measure for such spaces;[7] it is not our purpose in this paper to enter upon such questions.

Instead of projecting the surface points of $z = f(x, y)$ upon the XY plane, we may project them upon the X-axis and thus obtain other properties. For example, let us define the relationship $\mathfrak{R}_{r_1 r_2 r_3 r_4}$—between I-regions and points of the X-axis—as follows: $I\ \mathfrak{R}_{r_1 r_2 r_3 r_4}\ \xi$ if the surface points having x-coördinates in I have a limit point in the rectangle $x = \xi$ $r_1 \leqq y \leqq r_2$, $r_3 \leqq z \leqq r_4$. $\mathfrak{R}_{r_1 r_2 r_3 r_4}$ is closed. By applying Lemma II to this closed relation, we obtain the following result: *Let $z = f(x, y)$ be any single- or many-valued function defined in the entire XY plane. Let ξ be a point of the X-axis of the following character: a surface point (ξ, η, ζ) and a partial non-vanishing (linear) neighborhood N_ϵ of ξ exist such that (ξ, η, ζ) is not a limit point of surface points having x-coördinates in N_ϵ. The totality of points ξ is of measure zero.*

Similar results may be obtained for other metric properties and also in

the case of the descriptive properties. In the case of a function of n-variables, we may project upon an $(n-1)$ space, an $(n-2)$ space, etc.

The proofs of the preceding theorems are contained in a paper that has been offered to the *Transactions of the American Mathematical Society*.

[1] *Ann. Math. Princeton*, **18**, 1917 (147).

[2] For the terminology cf. Denjoy, *J. Math., Paris*, ser. 7, **1**, 1915 (122–125).

[3] Cf. for example, Fréchet, *Rend. Circ. Math. Palermo*, **22**, 1907, p. 1; and Hausdorff, *Grundzüge der Mengenlehre*, 1914, p. 211.

[4] Hausdorff, l. c., p. 315.

[5] For the case where only measurable sets are admitted cf., for example, de la Vallée Poussin, *Cours d' Analyse*, **2**, 1912, p. 114. For the linear case of general (not necessarily measurable) sets cf. Blumberg, *Bull. Amer. Math. Soc.*, **25**, 1919 (350).

[6] Cf. Denjoy, *Bull. Soc. Math. France*, **43**, 1915 (165).

[7] In this connection cf. Gâteaux, *Ibid.*, **47**, 1919 (47).

GENERALIZED LIMITS IN GENERAL ANALYSIS

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It is well known that to each of the various methods for summing divergent series there corresponds an analogous method for summing divergent integrals. It is readily seen that similar methods may be used for obtaining types of generalized derivatives of a function at a point where the ordinary derivative fails to exist. Likewise, in any other case in Analysis where we wish to associate a limit with a variable that oscillates, we will naturally be led to make use of methods that have been tried out in the case of divergent series.

It would be manifestly poor economy of time and thought to elaborate for each of these special theories such fundamental results as are common to them all, if these results can be obtained in one central theory that includes all the others. According to a principle of generalization formulated by E. H. Moore and stated by him on several occasions,[1] the existence of such a general theory is implied by the analogies found among the various special theories. It is natural to designate this general theory as the theory of generalized limits in General Analysis.

It is the purpose of the present communication to illustrate the nature and usefulness of this theory by outlining the proof of a theorem in it which is a generalization of one of the important theorems in the theory of divergent series. This latter theorem is the Knopp-Schnee-Ford theorem[2] with regard to the equivalence of the Cesàro and Hölder means of order k

for summing divergent series. Our genreal theorem includes as special cases this theorem, the analogous theorem of Landau[3] for divergent integrals, and a further new theorem with regard to the equivalence of generalized derivatives of the Cesàro and Hölder type. It can also be extended to the case of multiple limits, so that it includes new theorems, analogous to those mentioned, with regard to multiple series, multiple integrals, and partial derivatives. As this latter extension involves more complicated formulas, it will not be dealt with in the present communication.

Following the terminology introduced by E. H. Moore,[4] we indicate the basis of our general theory as follows:

$$(\mathfrak{A}\,;\mathfrak{P}\,;\mathfrak{S}\,;\mathfrak{G}^{\text{on }\mathfrak{S}\text{ to }\mathfrak{A}}\,;\mathfrak{H}^{\text{on }\mathfrak{S}\text{ to }\mathfrak{A}}\,;\mathfrak{F}^{\text{on }\mathfrak{S}\text{ to }\mathfrak{A}}\,;\varphi_0\,;J^{\text{on }\mathfrak{G}\text{ to }\mathfrak{H}\,.\text{ on }\mathfrak{H}\text{ to }\mathfrak{F}}),$$

where $\mathfrak{A} \equiv [a]$ denotes the class of all real numbers a, $\mathfrak{P} \equiv [p]$ denotes a class of elements p, and $\mathfrak{S} \equiv [\sigma]$ denotes a class of sets σ of elements p of the range $\mathfrak{P}$; $\mathfrak{G} \equiv [\gamma]$, $\mathfrak{H} \equiv [\eta]$, and $\mathfrak{F} \equiv [\varphi]$ are three classes of functions γ, η, and φ, respectively, on $\mathfrak{S}$ to $\mathfrak{A}$ (we shall restrict ourselves throughout to the consideration of single-valued functions); φ_0 is a special function φ of the class $\mathfrak{F}$; and J is a function on $\mathfrak{G}$ to $\mathfrak{H}$ and on $\mathfrak{H}$ to $\mathfrak{F}$, that is, a functional transformation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}$ or a function of the class $\mathfrak{H}$ into a function of the class $\mathfrak{F}$, denoted by $J\gamma$ or $J\eta$.

In order to make clear the relationship of our general theorem to the two special theorems referred to above, we will indicate here what the general basis reduces to in the particular instances III and IV:

$$\mathfrak{P}^{\text{III}} \equiv [\text{all } n = 1, 2, \ldots\ldots]; \quad \mathfrak{S} \equiv [\sigma_n \equiv (1, 2, \ldots n) \,|\, n];$$
$$\mathfrak{G} \equiv \mathfrak{H} \equiv \mathfrak{F} \equiv [\text{all } \gamma, \eta, \varphi \text{ on } \mathfrak{S} \text{ to } \mathfrak{A}];$$
$$\varphi_0(\sigma_n) = n \ (n); \quad (J\gamma)\ (\sigma_n) = \gamma(\sigma_1) + \gamma(\sigma_1) + \ldots. + \gamma(\sigma_n) \ (n).$$

$$\mathfrak{P}^{\text{IV}} \equiv [\text{all } a>0]; \quad \mathfrak{S} \equiv [\sigma \equiv (\text{all } x,\ 0<x<a) \ (a>0)];$$
$$\mathfrak{G} \equiv [\text{all functions that are finite and integrable (Lebesgue) on every finite interval } (0, a)];$$
$$\mathfrak{H} \equiv [\text{all } \eta = \textstyle\int_0^x \gamma \,|\, x>0)]; \quad \mathfrak{F} \equiv [\text{all } \varphi = (\textstyle\int_0^x \eta \,|\, x>0)];$$
$$\varphi_0(\sigma_a) = a \ (a); \ (J\gamma)\ (\sigma_a) = \textstyle\int_0^a \gamma\ (\gamma a); \ (J\eta)\ (\sigma_a) = \textstyle\int_0^a \eta\ (\eta a).$$

We next proceed to make certain postulates with regard to the nature of the elements in our basis, readily seen to be verified in the specific instances cited. Thus we require the class $\mathfrak{G}$ to have the linear property (L) as defined by E. H. Moore,[4] and the property (P) defined by

$$(P) \qquad \gamma_1 \cdot \gamma_2 \cdot \supset \cdot \gamma_1\gamma_2 \mathfrak{G},$$

the arguments of the two γ's being the same. It will then follow that $\gamma_P = \gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_n$ is of the class $\mathfrak{G}$. We require the class $\mathfrak{H}$ to have the properties (L) and (P), and the further property of being a sub-class of the

$\mathfrak{G}$, which property we shall denote by S_G. We require the class $\mathfrak{F}$ to have the properties (L) and (P), and the further property of being a sub-class of the class $\mathfrak{H}$, which property we shall designate as S_H. Hence is J also on $\mathfrak{F}$ to $\mathfrak{F}$.

We require the operation J to have the linear property (L) as defined by E. H. Moore, and the properties M_1 and M_2, defined as follows:

$$(M_1)\quad a_1<\gamma_1<a_2\cdot 0 \leqq \gamma_2\cdot\gamma_3 = \gamma_1\gamma_2\cdot \supset \cdot a_1(J_{\gamma_3})(\sigma) \leqq (J_{\gamma_3})(\sigma) \leqq a_2(J_{\gamma_3})(\sigma)(\sigma),$$

$$(M_2)\quad a_1<\gamma_1<a_2\cdot 0 \leqq \gamma_3\cdot\gamma_3 = \gamma_1\gamma_2\cdot\sigma''\supset\sigma'\cdot\supset\cdot a_1[(J_{\gamma_3})(\sigma'')-(J_{\gamma_3})(\sigma')]$$
$$\leqq (J_{\gamma_3})(\sigma'')-(J_{\gamma_3})(\sigma')$$
$$\leqq a_2[(J_{\gamma_3})(\sigma'')-(J_{\gamma_3})(\sigma')].$$

For the explanation of the other postulates we need certain auxiliary definitions. By the notation $\sigma' < \sigma''$, $\sigma'' > \sigma'$, we shall mean that σ'' contains all the elements of σ' and at least one element not found in σ'. For a given θ on $\mathfrak{S}$ to $\mathfrak{A}$, a given a, and a given σ' such that there exists $\sigma<\sigma'$, we shall write

$$\lim_{\sigma|\sigma<\sigma'} \theta(\sigma) = a$$

in the case that, corresponding to an arbitrary positive number e, there exists a $\sigma_e<\sigma'$ such that for every σ having the property $\sigma_e \subset \sigma<\sigma'$, $|\theta(\sigma) - a|<e$, or in symbols

$$e\supset :: \exists\, \sigma_e<\sigma', \sigma_e \subset \sigma<\sigma'\cdot\supset\cdot|\theta(\sigma)-a|<e.$$

This latter definition is based on the fundamental definition of limit in General Analysis given by E. H. Moore in the paper in these PROCEEDINGS referred to in the first foot-note.

We now postulate for the class $\mathfrak{S}$ the following properties:

$$(\mathsf{U})^5\quad \text{Either } \sigma'\cdot\supset\cdot\exists\,\mathsf{U}[\sigma<\sigma']^{\mathfrak{S}} \text{ or } \exists\,\sigma_0,\ \sigma'>\sigma_0\cdot\supset\cdot\exists\mathsf{U}[\sigma<\sigma']^{\mathfrak{S}},$$

$$(\mathsf{\Pi})^5\quad \sigma'\cdot\supset\cdot\exists\,\mathsf{\Pi}[\sigma>\sigma'].$$

In the typical instances in view in the formation of this general theory, of the two alternatives in (U) one holds and the other does not hold; however it is not assumed that this disjunction between the two alternatives shall be presupposed. In order to avoid notational reference to these alternatives, it is convenient to introduce a property—of sets σ of $\mathfrak{S}$; if the first alternative holds, every σ of $\mathfrak{S}$ has the property—, in notation $\bar{\sigma}$; if the first alternative does not hold, the sets $\bar{\sigma}$ are the sets $\sigma>\sigma_0$; further for brevity a property $\cdot$ (the negation of $-$) is used; thus every σ is a $\bar{\sigma}$ or a $\dot{\sigma}$.

We define

$$\sigma'_{-1} \equiv \mathsf{U}[\sigma<\sigma'] \quad (\bar{\sigma}),$$
$$\sigma'_1 \equiv \mathsf{\Pi}[\sigma>\sigma'],\ \sigma'_{n+1} \equiv \mathsf{\Pi}[\sigma>\sigma'_n] \quad (\sigma|n=1,2,3\cdots),$$
$$\bar{\varphi}(\sigma)\equiv\varphi(\sigma_{-1}) \quad (\bar{\sigma}),\quad \bar{\varphi}(\sigma)\equiv 0 \quad (\dot{\sigma}),$$

and with regard to the functions $\bar{\varphi}$ we postulate

(F) all functions $\bar{\varphi}$ are of the class $\mathfrak{F}$.

We also define the notation

$$(1) \qquad (D\theta)\ (\sigma) = \alpha(\sigma)$$

with regard to every θ, α on $\mathfrak{S}$ to $\mathfrak{A}$, to mean that

$$\lim_{\sigma|\sigma<\sigma'} \frac{\theta(\sigma') - \theta(\sigma)}{\varphi_0(\sigma') - \varphi_0(\sigma)} = \alpha(\sigma')\ (\bar{\sigma}),\ \theta(\sigma) = \alpha(\sigma)\ (\sigma).$$

We further postulate as to the class $\mathfrak{S}$

$$(R) \qquad \sigma' \cdot \supset \cdot \exists\ (\sigma'_1)_{-1} = \sigma'\ (\sigma'),$$

for the class $\mathfrak{H}$

$$(C) \qquad \eta \cdot \sigma' \cdot \supset \cdot \exists \lim_{\sigma|\sigma<\sigma'} \eta(\sigma) = \bar{\eta}(\sigma')\ (\bar{\sigma}'),$$

and for the class $\mathfrak{F}$

$$(\Delta) \qquad \exists\ D\varphi \equiv [(D\varphi)\ (\sigma)\ |\sigma]^{\mathfrak{H}}.$$

For the operation J we postulate

$$(I_D) \qquad \eta \cdot \supset \cdot (D(J\eta))\ (\sigma) = \eta(\sigma)\ (\sigma),$$
$$(I_J) \qquad \varphi \cdot \supset \cdot (J(D\varphi))\ (\sigma) = \varphi(\sigma)\ (\sigma).$$

We wish finally to postulate certain properties of the special function φ_0. We need first to introduce a further definition. We shall mean by the notation

$$\lim_{\sigma} \eta(\sigma) = \infty$$

that for every positive e there exists a σ_e (dependent on e) such that for every $\sigma > \sigma_e$, $\eta(\sigma) > e$, or in symbols

$$e : \supset : \exists \sigma_e\ ,\ \sigma > \sigma_e \cdot \supset \cdot \eta(\sigma) > e.$$

This definition also is based on E. H. Moore's definition of a limit referred to above. We also introduce for the sake of brevity the following notations

$$(2) \qquad \varphi_{0n}(\sigma) = \varphi_0(\sigma) \cdot \varphi_0(\sigma_1) \cdot \varphi_0(\sigma_2) \cdot \cdots \cdot \varphi_0(\sigma_{n-1})\ (n>1),\ \varphi_{01}(\sigma) = \varphi_0(\sigma),$$

$$(3) \qquad \psi_n(\sigma_1) = \frac{1}{\varphi_{0,n-1}\ (\sigma_1)}\ (n>1).$$

We now postulate for φ_0 that

(I) φ_0 is a real and positive function of σ,

(II) $\lim_{\sigma|\sigma<\sigma_n} [\varphi_0(\sigma_n') - \varphi_0(\sigma)] = 1 \quad (\sigma' n),$

(III) $\dfrac{1}{\varphi_{0n}(\sigma)} (J^n\eta)(\sigma)$ as function of σ is of the class $\mathfrak{F}(n)$,

the symbol $J^n\eta$ indicating that the operation J has been repeated n times

(IV) $\varphi_0(\sigma_n)$ as a function of σ is of the class $\mathfrak{F}(n)$,

(V) $\lim_{\sigma} \varphi_{0n}(\sigma) = \infty \quad (n)$

(VI) $(D\varphi_{0n})(\sigma) = n\varphi_{0,n-1}(\sigma) > 0 \quad (\sigma, n>1), \quad (D\varphi_{01})(\sigma) = 1 \quad (\sigma),$

(VII) $(D\psi_n)(\sigma_1) = -\dfrac{n-1}{\varphi_{0n}(\sigma)} \quad (\sigma, n>1).$

We have then as the foundation of our theory:

$$\Sigma \equiv (\mathfrak{A}; \mathfrak{P}; \mathfrak{S}; \mathfrak{G}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}}, LP; \mathfrak{H}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}}. LPS_GC;$$
$$\mathfrak{F}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}} LPS_H\Delta; {}_{\varphi_0}\mathfrak{F}. \text{ I II III IV V VI VII};$$
$$J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}. \text{ on } \mathfrak{H} \text{ to } \mathfrak{F}}. LM_1M_3I_DI_J).$$

We can now readily prove the two following properties of the operations D and J as applied to the class $\mathfrak{F}$:

(4) $(D[\varphi_1\varphi_2])(\sigma) = \varphi_2(\sigma)(D\varphi_1)(\sigma) + \bar{\varphi}_1(\sigma)(D\varphi_2)(\sigma) \quad (\sigma),$

(5) $(J[\varphi_2(D\varphi_1)])(\sigma) = \varphi_1(\sigma)\varphi_2(\sigma) - (J[\bar{\varphi}_1(D\varphi_2)])(\sigma) \quad (\sigma).$

We next formulate a further limit definition that is also based on the definition of E. H. Moore to which reference has already been made. We shall say that $\eta(\sigma)$ approaches a limit a as to σ if, corresponding to every positive e, we can find a σ_e such that for $\sigma > \sigma_e$ we have $|\eta(\sigma) - a| < e$, or in symbols

(6) $\lim_{\sigma} \eta(\sigma) = a :\cdot \equiv :e: \supset : \exists \sigma_e, \sigma > \sigma_e . \supset . |\eta(\sigma) - a| < e.$

We are now ready to define the two generalized limits with which we are concerned. Given any function $\eta(\sigma)$, we set

(7) $(C_n\eta)(\sigma) = [n!/\varphi_{0n}(\sigma)](J^n\eta)(\sigma) \equiv \dfrac{n!}{\varphi_0(\sigma)\,\varphi_0(\sigma_1)\cdots\varphi_0(\sigma_{n-1})} (J^n\eta)(\sigma),$

(8) $(M\eta)(\sigma) = [1/\varphi_0(\sigma)](J\eta)(\sigma),$

(9) $(H_n\eta)(\sigma) = (M^n\eta)(\sigma),$

where C and H are used, as is customary, to connote Cesàro and Hölder. If for a fixed n $\lim_{\sigma} (C_n\eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Cn) for $\eta(\sigma)$. If $\lim_{\sigma} (H_n\eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Hn) for $\eta(\sigma)$.

We can now prove the equivalence of these two generalized limits by a method similar to that devised by Schur[6] for proving the special theorem in the case of series. The proof breaks up into three lemmas and a theorem.

LEMMA 1. *If we represent by E the identical functional operation, $E\theta = \theta(\theta)$, we have the identity*

$$(10) \qquad \left(\left(\tfrac{n-1}{n}M + \tfrac{1}{n}E\right)(C_n\eta)\right)(\sigma) = (M(C_{n-1}\eta))(\sigma).$$

LEMMA 2. $\mathfrak{H}\lim_{\sigma}\varphi(\sigma) = a \cdot |\varphi(\sigma)| < a_1(\sigma) \cdot \supset \cdot \mathfrak{H}\lim_{\sigma}[\varphi_{0n}(\sigma)]^{-1}(J\varphi_n)(\sigma) = \frac{a}{n}$,

where

$$\varphi_n(\sigma) = \varphi_{0,n-1}(\sigma)\varphi(\sigma)\ (n \geqq 2), \quad \varphi_1(\sigma) = \varphi(\sigma),$$

$\varphi_{0n}(\sigma)$ *being defined by* (2).

LEMMA 3. *If we set*

$$\omega(\sigma) = \tfrac{1}{n}\varphi(\sigma) + \tfrac{n-1}{n} \cdot \frac{1}{\varphi_0(\sigma)}(J\varphi)(\sigma),$$

$$\mathfrak{H}\lim_{\sigma}\omega(\sigma) = a \cdot \supset \cdot \mathfrak{H}\lim_{\sigma}\varphi(\sigma) = a.$$

Theorem. $\mathfrak{H}\lim_{\sigma}(C_n\eta)(\sigma) = a \cdot \sim \cdot \mathfrak{H}\lim_{\sigma}(H_n\eta)(\sigma) = a.$

[1] E. H. Moore, *Introduction to a Form of General Analysis*, The New Haven Mathematical Colloquium, Yale Univ. Press, 1910, p. 3; "On a Form of General Analysis with Applications to Linear Differential and Integral Equations," *Atti IV Cong. Inter. Mat.* (Roma, 1908), **2**, p. 98; "On the Foundations of the Theory of Linear Integral Equations," *Bull. Amer. Math. Soc.*, Ser. 2, **18**, p. 339; "Definition of Limit in General Integral Analysis," these PROCEEDINGS, **1**, 1915, p. 628.

[2] K. Knopp, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze," *Inauguraldissertation* (Berlin, 1907); W. Schnee, "Die Identität des Cesàroschen und Hölderschen Grenzwertes," *Math. Annalen*, **67**, 1909, pp. 110–125; W. B. Ford, "On the Relation between the Sum-Formulas of Hölder and Cesàro," *Amer. J. Math.*, **32**, 1910, pp. 315–326.

[3] E. Landau, "Die Identität des Cesàroschen und Hölderschen Grenzwertes für Integrale," *Leipzig Berichte*, **65**, 1913, pp. 131–138.

[4] Cf. the first three references in the first foot-note; also E. H. Moore, "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations," *Proc. Fifth Inter. Congr Math.* (Cambridge, 1912), **1**, 1913, pp. 230–255.

[5] For the definition of these symbols see page 150 of the monograph referred to in the first footnote.

[6] I. Schur, "Über die Äquivalenz der Cesàroschen und Hölderschen Mittelwerte," *Math. Annalen*, **74**, 1913, pp. 447–458.

A PRELIMINARY NOTE ON TETRAMITUS, A STAGE IN THE LIFE CYCLE OF A COPROZOIC AMOEBA

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In 1920 a study of the Entamoeba of the rat was begun and in a culture on artificial medium a number of Coprozoic amoeba appeared, one of which exhibited a flagellate phase which seems to be identical with Tetramitus rostratus Perty.[1]

The appearance of flagellate phases in cultures of Coprozoic amoeba (Whitmore,[2] etc.) as well as of soil amoeba (Kofoid,[3] Wilson,[4] etc.) has been recorded a number of times, but the flagellates which appeared were relatively simple in organization and very transitory in occurrence. Furthermore, some of the simpler flagellates have been observed (Pascher[5]) to lose their flagella and become amoeboid. In the case here described, however, there is a transformation of a simple amoeba into a relatively complex flagellate which multiplies for several days then returns to the amoeboid condition and becomes encysted. Tetramitus rostratus has been studied by a number of investigators since its discovery in 1852, by Perty,[1] yet no one has given a full account of its life history. The lack of cysts in previous accounts, mentioned by Dobell and O'Connor,[6] is explained by the transformation into an amoeba before encystment. It is believed that this animal presents the extreme in transformations of this sort and serves to emphasize the close relationship between amoebae and flagellates, and the need for careful studies of life cycles, in pure cultures where possible, in investigations of coprozoic and other Protozoa. It is proposed to record at this time a brief account of the life history of this amoebo-flagellate reserving the more extended discussion for future publication.

The amoeba in question was first observed on November 5, 1920 in coecal material which had been taken from the rat and placed in sterile physiological salt solution on October 28, 1920. This is known as culture 10. After discovery of the amoeba, transplants were immediately made in various media. The amoebae were subcultured several times on solid media over a period of about two months, then two cultures were made in fluid media. In one the formula given by Sellards,[7] namely, 1.0 gr. Witte's peptone, 0.5 gr. lactose, 1000 cc. distilled water was used; in another, dextrose was substituted for lactose in Sellards' formula. The medium containing dextrose proved to be much more favorable for the amoeba under consideration. In these liquid media a flagellate appeared which is thought to be Tetramitus rostratus Perty.

In these cultures differences among the active amoebae as well as three types of cysts and the flagellate were noted, hence pure lines were started by the isolation method. Pure lines derived from culture 10 were designated G., I., J. and L.; of these G. and L. are of more special interest, the former being obtained from a single specimen of Tetramitus, the latter from a smooth, thin-walled cyst.

The cysts of cultures G. and L. revealed no microscopic differences either in the living or in the fixed and stained condition, and when transplants were made of the cysts in favorable media both active amoebae of limax type and Tetramitus developed. Pure lines started from an isolated amoeba, an isolated flagellate or an isolated cyst in liquid or semi-liquid media at room temperature have always given the same results, i.e., smooth, thin-walled cysts, the active amoeba of limax type and Tetramitus rostratus.

Although the amoeba with the Tetramitus stage was obtained in cultures from the coecal contents of the rat, the presence of contractile vacuoles etc., indicated that this amoeba was not a parasite but belonged to the coprozoic protozoa as defined by Dobell and O'Connor.[6] Feeding experiments upon rats and mice were conducted to determine whether the amoeba and flagellate stages are present either in the small intestine or in the coecum of these animals. Prior to feeding these experimental animals their faeces was collected on filter paper and cultures made in liquid media; in a few cases the amoeba and tetramitus stage was already present. The nine rats and four mice used were killed and opened in one, two, three, and five days after feeding and microscopic examination made of the fresh contents of the coecum and small intestine. Neither active amoebae nor Tetramitus were found, but cysts, both degenerate and normal, were present in all the animals. Cultures were made of coecal contents in sterile liquid media which were studied the following day. The active amoeba of the type used in the feeding and Tetramitus were found in these cultures. The results of the feeding experiments indicate that the amoeba under cultivation belongs to the coprozoic Protozoa.

Much evidence that Tetramitus is a stage in the life history of this coprozoic amoeba was obtained from the study of the living animal and from fixed and stained cover glass preparations, yet to affirm this fact with certainty, many consecutive hours of many days have been spent in close observation. Much of this study has been conducted on hanging drops, made either from a single cyst isolated and placed in a drop of fresh liquid medium or a few cysts taken by a platinum loop from a culture containing only cysts and adding from two to three loops of fresh medium. All were placed on depression slides and ringed with hard vaseline. By this method the animals have been kept under observation during the entire

life cycle, a period of about four days. A brief description of the living animal during the stages of its life cycle will now be given.

1. *The Cyst:*—(fig. 1, A) *a.* Morphology. Commonly spherical, occasionally broadly oval, single, smooth, relatively thin wall; usually uninucleate; rarely binucleate; size ranging from 6μ to 18μ.

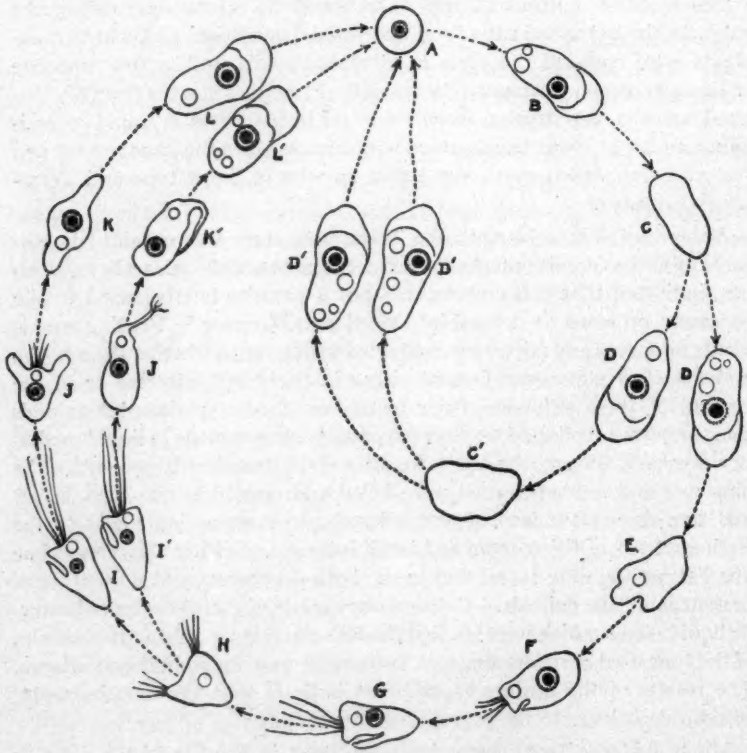

FIGURE 1

Diagram of life-cycle of coprozoic amoeba with Tetramitus as its flagellate phase. A—the cyst; B—vegetative amoeba; C, C'—division ("gel" state); D, D'—amoebae after division; E, F—phases in transformation of Tetramitus; G—fully formed Tetramitus; H—Tetramitus previous to division ("gel" state); I, I'—Tetramitus after division; J, K, L and J', K', L'—series of transformation stages, Tetramitus to amoeba.

b. Excystation. No empty cyst wall remains after the amoeba has emerged from the cyst, the wall apparently undergoing solution during the process of excystation.

2. *Amoeba:*—(fig. 1, B) *a.* Vegetative stage. Up to the present date the animal has always been observed to leave the cyst stage as an amoeba, never as a Tetramitus. The amoeba varies from 14μ to 48μ in length.

The rounded vesicular nucleus is usually 5μ to 6μ in diameter, and contains a caryosome 1μ to 2μ less in diameter; the pericaryosomal zone is usually clear but may contain a few granules. Peripheral chromatin on the nuclear wall in the form of granules can sometimes be observed in the living condition and generally in the fixed and stained specimens. The cytoplasm becomes sharply defined into a non-granular ectoplasm and granular endoplasm during the protrusion of pseudopodia. The amoeba generally moves by means of one broadly lobose pseudopodium, which however, may not be withdrawn before additional ones are formed. There is always one large contractile vacuole, augmented frequently by a few smaller ones.

b. Division. Gradually the rapid movement of the granules and of the nucleus through the endoplasm becomes slower, the pseudopodium is retracted, the granules clump together, then lose their identity so that the entire protoplasmic body becomes homogeneously refractive, and thus a "gel" state is formed during which the nucleus is obscured (fig. 1, C). The animal may remain quiescent in this state for from thirty seconds to several minutes, after which it suddenly elongates, constricts in the middle and divides into two daughter cells, meanwhile passing from the "gel" to the "sol" state (fig. 1, D, D'). The formation of a "gel" state during the process of division in higher animals has been noted in many papers by a number of observers, among whom are Chambers[8] and Heilbrunn.[9] Multiplication may continue for a number of generations without flagellation, or the amoeba may pass through a flagellate phase. Finally in each case the amoeba becomes encysted.

3. *Flagellate Phase:*—(fig. 1, G). Tetramitus varies much in size and in form as has been noted by Perty,[1] Fresenius,[10] Dallinger and Drysdale,[11] Stein,[12] and Klebs,[13] but the typical form is that described by them as "conical" with the pointed end directed posteriorly. The contractile vacuole and cytostome are located at the flattened anterior end which bears four flagella. The nucleus is slightly posterior to the contractile vacuole, and is similar in organization to that of the amoeba just described except that it is smaller measuring 3μ to 5μ in diameter. The animal is usually 14μ to 18μ in length and 7μ to 10μ in greatest width.

4. *Transformation of an Amoeba to Tetramitus:*—After the amoeba has passed through one or more generations by division, some of them may transform into flagellates. While the morphological changes may vary during the process, the results of one observation will now be given in detail. An amoeba retracted its pseudopodium, but changes in outline

continued slowly; the cytoplasm became homogeneously refractive and the nucleus was obscured probably due to a "gel" state (fig. 1, E). The contractile vacuole continued to pulsate slowly; an involution of the lateral margin near this organelle was formed which terminated in the protoplasmic body as a cleft. The animal now assumed an elongated somewhat pyriform shape, the margin near the vacuole being flattened. From this area flagellar activity began, and thus it became oriented as the anterior end. The nucleus, nearer the posterior end of the animal became demarcated from the homogeneously refractive cytoplasm (fig. 1, F). The animal began to move rapidly, undergoing many contortions, during which the cytoplasm became resolved into refractive granules suspended in a clear substance, the nucleus moved towards the anterior end and the pellicle became more firm so that the flagellate shape was assumed (fig. 1, G). The flagella had reached the average length and the characteristic rotation around the longitudinal axis began as the animal swam swiftly about.

6. *Division of the Flagellate:*—In a short time after the above described change the animal suddenly became broadly triangular in form (fig. 1, H), flagellar activity being observed at each angle limiting the anterior end. The animal then passed into the "gel" state and this was followed by longitudinal division (fig. 1, I, I′). The entire transformation, followed by fission in the above case, was completed in one hour, but these processes are not always so rapid. The flagellate may continue to multiply for several hours or even days commonly dividing into two by longitudinal fission but occasionally into three or four equal or unequal offspring.

7. *Transformation of Flagellate into an Amoeba:*—Eventually the Tetramitus transforms into an amoeba unless death prevents. The morphological changes which form this sequence are not always the same, but one set of observations will be given in detail. A Tetramitus suddenly shortened and became broadly triangular, two short processes were protruded from the angles of the flattened anterior end (fig. 1, J). Soon the body became rounded, the processes longer (fig. 1, K), and the movements slower resulting in but slight changes of location. Meanwhile the protoplasm exhibited external form changes marked by contortions of the body. The successive changes were very rapid and suggested that an effort was being made by the animal to free itself of flagella. Finally it became quiescent a few moments and then changed into the form of the typical amoeba which progressed by a pseudopodium of lobose type instead of by flagella (fig. I, L). Another series of changes are indicated by fig. 1, J′, K′ and L′. The transformation process is sometimes prolonged through many hours. The amoebae thus derived may multiply by division but ultimately they become encysted completing the life cycle.

The time periods occupied by the different stages in the life cycle at room temperature, as a rule, have been the following:—The cyst lives in a proper medium indefinitely. When cysts are planted on fresh media, amoebae may emerge in two or three hours and remain active from three to four days, rarely longer, and then encyst. The Tetramitus may appear in ten to twelve hours after transplantation of the cysts, continue numerous for two days and gradually disappear the third day as these become transformed into amoebae or die.

Exceptions to the above have occurred in two cultures in which activity of Tetramitus lasted for about a month. The study of the living animals together with the cytological examination of fixed and stained specimens from each culture revealed an increase in the number of nuclei to two, three and four, also the presence of giants and a greater proportion of animals of irregular shape.

While great numbers of these flagellates have been under observation for more than a year, no stages in the life history have been observed other than those described above. Binary fission for Tetramitus has been noted by Perty,[1] Dallinger and Drysdale,[11] Stein,[12] Klebs,[13] and Alexeieff.[14] Although no sexual stages nor spore formation such as described by Dallinger and Drysdale have been observed in these studies, nevertheless these observers did see the flagellate become partly amoeboid and a longer observation should have revealed to them the complete change to an amoeba.

In a life cycle of this kind, in the absence of sexual phenomena it is uncertain whether the amoeboid or the flagellate phase should be considered as the adult or dominant one, especially since non-sexual reproduction is extensive in both phases. A thorough study of the life cycles of other "amoebae" and other "flagellates" might reveal further examples of transformations of the kind here described.

In closing I wish to express my great indebtedness to Doctor D. H. Wenrich who has been ever ready to aid this investigation by participation and by fruitful suggestions. Also for the continued interest and aid of Professor Clarence E. McClung who has helped to surmount the many difficulties encountered and who, with the Graduate School of the University of Pennsylvania, has provided the facilities of the Zoölogical Laboratory for this research.

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A FURTHER NOTE ON THE AGE INDEX OF A POPULATION[1]

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In an earlier paper in these PROCEEDINGS, Pearl[2] proposed the function

$$\phi = S\left\{\frac{\Delta^2}{P}\right\}(M - M_\phi)$$

as a single numerical index of the age distribution of a population. In this expression S indicates summation, for all age groups given in the original statistics, of the expression in brackets; Δ is the deviation, for each age group, of the percentage of the actual population in each age group from the percentage of the same group in the standard population of Glovers[3] Life Table, denoted by P; M = mean age of living population; M_ϕ = mean age of persons in a stationary population unaffected by migration.

In connection with certain problems now under investigation in this laboratory it becomes a matter of importance to know how sensitive this age index is to change in the size of the age groups of the original statistics. Since its original proposal the index has been used in many studies in this laboratory, and the larger experience has strengthened our confidence in its reliability as an index of significant variations in the age constitutions of populations. But it has always been used hitherto with at least 6 to 8 age groups covering the life span. Suppose the original statistics furnish only 3 age classes for the entire life span. Will this age index ϕ then give a reliable picture of the significant variations in age distribution, as we pass from city to city, or county to county?

To test this point, the obvious thing to do is to determine the correlation between the age indices for n communities, on the basis of say 3 divisions of the life span, with the age indices for the same communities on the basis of say 6 divisions of the life span. If the correlation is high it will mean

that even with the coarse grouping the index is reflecting the main differences in the form of the age distributions.

The present note reports a test of this sort. The 1915 census of Iowa[4] gives in Table I the age distribution of the inhabitants of each of the 99 counties of the state in such form as to permit the formation of the follow-

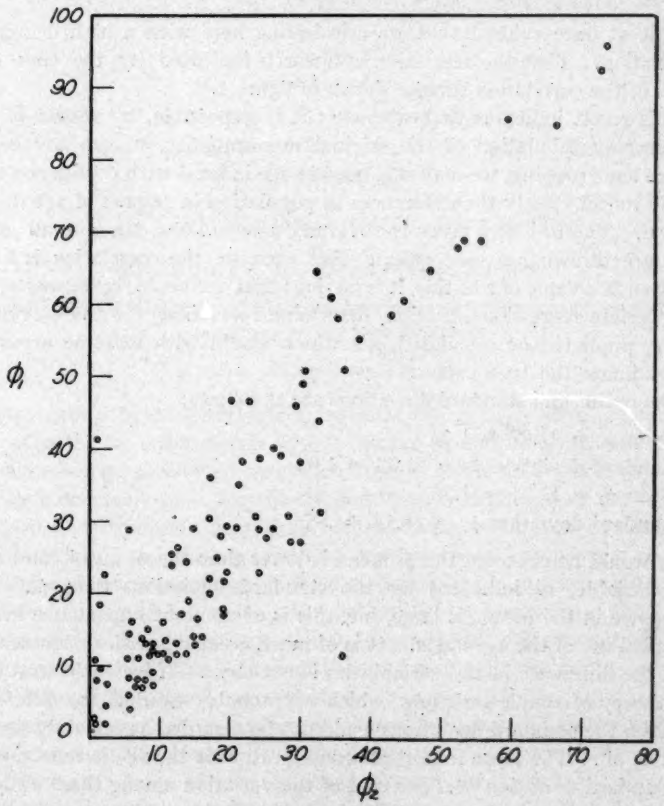

FIGURE 1

Spot diagram showing the correlation between ϕ_1 and ϕ_2. Each dot represents one county.

ing classes: 0–4, 5–9, 10–17, 18–20, 21–44, 45 and over, both figures being inclusive in each case. For each of the 99 counties ϕ was computed on the basis of this grouping of the ages. Call these values ϕ_1. Then for the same 99 counties the population was combined by addition into the 3 age groups 0–9, 10–20 and 21 and over. A new set of ϕ's was then calcu-

lated on the basis of this coarse grouping. Call these values ϕ_2. Then what we wish to know is the value of the correlation $r_{\phi_1\phi_2}$.

This was calculated by the direct product moment-method, without grouping and gave as a result

$$r_{\phi_1\phi_2} = + .84 \pm .02.$$

It is at once evident that we are dealing here with a high degree of correlation. That the regression is linear is indicated by the spot diagram of the correlation surface shown in figure 1.

This result indicates that whenever it is impossible, by reason of unsatisfactory tabulation of the original raw statistics, to get any but a coarse age grouping, we may still use the age index ϕ with confidence that it will reflect closely the differences in population in respect of age distribution. The test here given is obviously a severe one, because all adult ages are thrown into one group. But even so, the correlation is high. With such a value of r as this, it is obvious that one could, by means of the appropriate regression equation, determine corrections for the ϕ_2 values, giving predicted or calculated ϕ_1's which would with extreme accuracy approximate the true values.

The means and standard deviations are as follows:

Mean $\phi_1 = 27.27 \pm 1.46$
Standard deviation $\phi_1 = 21.53 \pm 1.03$
Mean $\phi_2 = 19.71 \pm 1.23$
Standard deviation $\phi_2 = 18.14 \pm .87$

As would be expected the ϕ_2 mean is lower than the ϕ_1 mean, and ϕ_2 is less variable, as indicated by the standard deviation, than ϕ_1. The difference in the means is large, but this is of no great importance in the practical use of the age index. It is of much greater practical significance that the difference in the variabilities is not large. If we could trust here the theory of simple sampling (which we probably cannot) the difference between the standard deviations could not be regarded as certainly significant at all. The more important consideration is that ϕ_2 is reflecting in its standard deviation 84.2 per cent of the variation among these 99 Iowa counties in respect of age distribution of their populations, as measured by ϕ_1.

In conclusion it may be stated that since, as is demonstrated in this paper, there is a high correlation between the values of ϕ for extremely coarse age groupings, with the values obtained from finer groupings, this age index may be used with considerable confidence in cases where data on age distribution are available only for a few broad classes covering the life span. All our experience with this age index ϕ since its publication two years ago has confirmed our belief that it is a function of great value to the vital

statistician who wishes to make analytical studies of medical and hygienic data by modern statistical methods.

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THE THERMAL IONIZATION OF GASEOUS ELEMENTS AT HIGH TEMPERATURES

A CONFIRMATION OF THE SAHA THEORY

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Read before the Academy, April 25, 1922

In a recent article Eggert[1] has shown that the ordinary thermodynamic expression for the effect of temperature on the equilibrium of chemical reactions, when supplemented by assumptions as to the values of the special constants occurring in it, can be applied to the calculation of the extent to which under different conditions neutral atoms of the gaseous elements are converted into positive ions and free electrons. Saha[2] has shown that the most uncertain factor in Eggert's formulation can be eliminated by introducing the ionization-potential of the element. He has then calculated the dissociation of the first electron from the atoms of many important elements, thus, the extent to which such reactions as $Na = Na^+ + E^-$ take place, where E^- represents electron gas. He has tabulated the values of this percentage ionization at various temperatures and pressures; and he has especially pointed out the great significance which these considerations may have for the interpretation of the spectra of elements under solar and stellar conditions.

These principles are of great interest to astrophysicists, as may be illustrated by the following applications. It has long been known that the solar prominences show strongly the hydrogen and helium lines and the enhanced (H and K) lines of calcium, but not the blue line of calcium nor the familiar lines of sodium or of other alkali elements. The remarkable absence of these lines had not been explained; but the thermodynamic relations now show that the extremely small pressure combined with the high temperature of the prominences must greatly promote the

ionization of all elements, and may well suffice to convert the neutral calcium atoms into calcium ions, which produce the H and K lines; and also to convert completely the neutral sodium atoms, to which the D lines are due, into sodium ions, which do not yield strong lines in the visible spectrum. A second interesting application of Saha's equations has been made by Russell,[3] who was led to the discovery of rubidium in sun spots by the considerations that the absence of its lines in the general solar spectrum is probably due to the complete conversion by the high temperature there prevailing of the neutral rubidium atoms (which produce the characteristic lines of the element) into rubidium ions (Rb^+), and that at the lower temperature prevailing in sun spots this conversion might well be only partial.

The thermodynamic expressions may first be presented which are applicable to a reaction of the type under consideration, by which the neutral atoms of a gaseous elementary substance M are converted into positive ions and electrons in accordance with the chemical equation $M = M^+ + E^-$. At any definite temperature the equilibrium-constant K of such a reaction is expressed in terms of the partial pressures, p_M, p_{M^+}, p_{E^-}, of the three substances, regarded as perfect gases, by the following equations:

$$\frac{p_{M^+}\, p_{E^-}}{p_M} = K \qquad (1) \qquad\qquad \frac{x^2 p}{1 - x} = K. \qquad (2)$$

Unlike the first equation, which is general, the second equation, in which x represents the fraction ionized and p the sum of the pressures p_M and p_{M^+}, is valid only when $p_{M^+} = p_{E^-}$; that is, only when electrons do not originate from any other source, as from the presence of another ionizing element or from thermionic causes.

The second law of thermodynamics leads to a simple expression for the change of this ionization constant K with the temperature, in the case where the heat-content-increase ΔH attending the reaction can be expressed as a linear function of the absolute temperature T, thus by the formula $\Delta H = \Delta H_0 + T\Delta C_p$, where ΔH_0 is an empirical constant and ΔC_p is the increase in the heat-capacity at constant pressure that results from the occurrence of the reaction. The second law leads, namely, to the following equations, in which R is the gas-constant (1.985 calories per degree), and I is the integration-constant resulting from the integration of equation (3).

$$d \ln K = \frac{\Delta H_0 + T\Delta C_p}{RT^2}\, dT. \qquad (3)$$

$$\ln K = -\frac{\Delta H_0}{RT} + \frac{\Delta C_p}{R} \ln T + \frac{I}{R}. \qquad (4)$$

Now Saha introduces the following assumptions as to the quantities occurring in this equation:

(1) That the energy increase attending the ionization of a single molecule is equal to the electrical work Ve that must be expended in order to give to an electron a kinetic energy which just suffices to ionize an atom of the element with which it collides; e being the charge on the electron and V the ionization-potential, which last may be determined by direct measurement or computed from the spectral series of the element.

(2) That the heat-capacities of the neutral atom M and its ion M^+ are equal, and that the heat-capacity of electron gas is the same as that of any other monatomic gas, namely, 2.5 R per mol of the gas.

(3) That the constant I, which is the sum $I_{M^+} + I_{E^-} - I_M$, of constants characteristic of the three substances (the positive ion, the electron, and the neutral atom), can be evaluated by assuming $I_{M^+} = I_M$ and by calculating I_{E^-}, the constant for electron gas, by a theoretical principle that has been shown applicable to ordinary monatomic gases,[4] which need not be here described.

These considerations lead to the following values of the constants occurring in equation (4):

$$\frac{\Delta H_0}{R} = \frac{N e V}{R} = \frac{96500\ V}{8.316} = 11625\ V; \quad \frac{\Delta C}{R} = 2.5; \quad \frac{I}{R} = -15.10.$$

Substituting these values in equation (4) and changing from natural to ordinary logarithms, we get

$$\log K = -\frac{5048\ V}{T} + 2.5 \log T - 6.56. \qquad (5)$$

By this equation the ionization-constant K of an element at any absolute temperature T can evidently be calculated from its ionization-potential V (in volts); and from this ionization-constant its actual ionization can then be found by equation (1) or (2).

The main purpose now of this paper (of which a much more complete form will soon appear in the *Journal of the American Chemical Society* and the *Astrophysical Journal*) is to point out that the earlier investigation of Arrhenius,[5] and especially the more recent researches of H. A. Wilson,[6] on the electrical conductivity of flames into which salt solutions are sprayed clearly indicate that the conduction arises from an ionization process of the type here considered, and that these measurements make possible a computation of the relative magnitudes of the ionization-constants. Thus the experiments of these investigators have established the following facts.

1. The conductance of the flame is independent of the acidic constituent

of the salt; thus it is the same for the chloride and carbonate of any given alkali element. This is in accord with the assumption that these salts are completely decomposed in the way expressed by the chemical equations $2\ MCl = 2\ M + Cl_2$, or $M_2CO_3 = 2\ M + CO_2 + {}^1/_2\ O_2$, and that the conductance is due to ionization of the metallic element M which is common to its various salts.

2. The conductance changes with the concentration of any given salt and in mixtures of salts in the way required by equations (1) and (2), showing that the ions result from an ionization process in which one molecule dissociates into two others, as is the case in the reaction $M = M^+ + E^-$, where E^- denotes electron gas.

3. There is present in the flames both positive and negative carriers, but the mobility of the latter has been shown to be roughly 2500 times as great as that of the former, clearly indicating that the negative carriers are electrons and that substantially the whole conductance is due to them.

From the way in which the conductance changes with the concentration it was found possible by a mathematical process that need not be here described to calculate under the above stated assumptions relative values of the ionization constants K of the five alkali elements whose salts were investigated by H. A. Wilson. These relative values could then be compared with the absolute values calculated from the ionization-potentials by the thermodynamic equation (equation 5 given above). The results are reproduced in the following table.

TABLE 1

VALUES OF THE IONIZATION-CONSTANT ($10^{12}\ K$) AT 2000° ABSOLUTE

	Cs	Rb	K	Na	Li
From flame conductances	3500	780	430	5.6	0.8
From ionization-potentials	8200	1600	620	6.2	1.4

It will be seen that the ratios of the values for the successive elements run closely parallel to each other in the two series. Especially noteworthy is the very large change in both series in passing from sodium to potassium, and the much smaller differences between the other elements.

As stated above, the flame conductance data lead only to relative values of the ionization-constants; and the values given in the preceding table would contain an arbitrary multiplying factor if they were based on those data alone. It was found possible, however, by combining with the conductance data the results of previous rough determinations of the mobility of the electrons[7] and of the number of them per cubic centimeter[8] in entirely similar flames free from salt, to obtain absolute values of the ionization constants; and these are the numbers that are recorded in the first row of the above table. Although they afford an estimate only of the order

of magnitude of the constants, yet they are seen to correspond well with the values calculated by the thermodynamic equation.

The results here presented therefore afford a striking confirmation of the general validity of the calculation of the ionization of elements from their ionization-potentials by the thermodynamic equation employed by Saha.

[1] Eggert, *Physik. Zs.*, **20**, 570, 1919.
[2] Saha, *Phil. Mag.*, **40**, 478, 809, 1909; **41**, 267, 1921.
[3] Russell, *Astrophys. J.*, **55**, 129, 1922.
[4] Sackur, *Ann. Physik.*, **36**, 598, 1911; **40**, 67, 1913; Tetrode, *Ibid.*, **38**, 434; **39**, 255, 1912; Stern, *Zs. Electrochem.*, **25**, 66–80, 1919; Tolman, *J. Amer. Chem. Soc.*, **42**, 1185, 1920; **43**, 1593, 1921; Lewis, *Physic Rev.*, **18**, 121, 1921.
[5] Arrhenius, *Wied. Ann. Phys.*, **43**, 18, 1891.
[6] H. A. Wilson, *Phil. Trans. Royal Soc.*, **A216**, 63–90, 1915.
[7] H. A. Wilson, *Phil. Mag.*, (6) **10**, 475, 1905.
[8] Wilson and Gold, *Phil. Mag.*, (6) **11**, 484, 1906.

AN ELECTRON THEORY OF ELECTRIC CONDUCTION IN METALS

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This is a modification and extension of the dual conception of conduction which I set forth in a paper[1] printed several years ago and to which I have referred in more recent publications. The theory in its present shape, like every other theory of metallic conduction with which I am acquainted, involves propositions that may well be questioned, but it contains certain features not present, so far as I know, in other theories, and I hope these will be found worthy of consideration, even if they are finally to be rejected.

I shall endeavor to state this theory, without extended argument, in the form of certain propositions, some of which are or involve outright assumptions. Any helpful hypothesis, not in conflict with well established facts or principles, I have considered justifiable, in view of the difficulties of the problem with which I am dealing.

Behavior of the Ions in Conduction

1. Complete metal atoms are of such hardness and symmetry that they are not subject to appreciable polarization or orientation in impressed electric fields of ordinary strength.

2. An ion is formed from an atom by loss of an electron from the outer shell and is, accordingly, unsymmetrical, essentially polarized, and therefor subject to orientation in a uniform electric field such as may be ap-

plied to a metal from without. This proposition seems to be in general accord with Langmuir's conception of atomic shells. See no. 991 in *Science Abstracts* for 1919.

3. The behavior of these ions in an electric field is somewhat like that of magnetic particles in a magnetic field, as set forth in Langevin's theory[2] of paramagnetism, or that of electric dipoles as described by Sir J. J. Thomson.[3] But in one vital particular, indicated in the next paragraph, the formula used by Langevin and by Thomson must be modified to suit the present case.

4. When and where the equipartition law of energy holds, the Boltzmann-Maxwell distribution law can be written with the factor $e^{-V/RT}$, where V is the mean potential energy of a particle with respect to any coördinate, either of position or of direction, in the field of force, and RT is twice the mean kinetic energy of the particles with respect to any degree of freedom whatever. But when and where the equipartition law fails, we must, if we can use the Boltzmann-Maxwell formula at all, use it with regard to the particular coördinate or degree of freedom that we are dealing with. Kinetic energy along one coördinate may in this case have nothing to do with the distribution of particles along another coordinate. For example, if all the kinetic energy of air particles became rotary energy, the atomosphere would fall. So, too, if we are dealing with particles that have no rotary energy, the translatory energy of these particles will have nothing to do with their orientation in a directing field of force.

5. The atoms and the ions have a mean kinetic energy of rotation proportional to the absolute temperature but of a much smaller order of magnitude than that of the energy of translatory vibration.[4] Accordingly we should, in the Boltzmann-Maxwell formula, use a very much smaller value than Langevin and Thomson have used in the denominator of the exponent of e, thus getting a much greater orientation, in a given field of force, than their formula would lead us to expect. This orientation will, through a great range of electric field strength, be proportional to the strength of the field and inversely proportional to the absolute temperature.

6. The time required for the orientation of an ion to reach its final value is large compared with the period of the translatory vibrations of the ion. See (21).

7. An ion has on its surface a fairly definite scar or *pit* from which an electron has been dislodged, and the ion can capture an approaching electron, thereby recovering the status of an atom, only when the electron strikes, not too violently, within this pit.

8. A free electron, approaching an ion with the kinetic energy of a gas molecule, may, though attracted toward the pit, shoot by, failing to make

connection, as a comet shoots by the sun. Accordingly, termination of the life of an ion by the taking up of a free electron may be a comparatively rare occurrence, most of the ions ending by taking up electrons directly from neighboring atoms.

9. Though the atoms have a regular arrangement in a continuous metal crystal, the individual crystals in any ordinary piece of metal are so small and so heterogeneously placed with respect to each other, that we may think of the atoms and ions as, on the whole, subject to mutual collision with any direction whatever of their line of centres. Accordingly we may think of the *pits* as *forming* anywhere on the ions, in one quarter as probably as in any other, when no impressed electric field is in operation.

10. Impressed electric fields of ordinary strength have no appreciable influence in determining or preventing the passage of an electron from an atom to a neighboring ion.[5] Thus a potential gradient of 1 volt per centimeter, imposed from without, would be a very steep one, whereas, if the "ionizing potential" of an atom within the solid metal is of the order of 1 volt, as I believe it to be, the field of force which a neighboring ion may exert on an electron is of the order of 10^8 volts per centimeter. It is possible, however, that when extremely great current densities are used, as in attempts to pass beyond the range of Ohm's law, the impressed field may be great enough to have a perceptible effect in determining whether an electron shall stay on its present atom or pass over to an adjacent ion. Such an effect would tend to make the current increase more rapidly than the potential gradient and might do something to mask the opposite kind of change that would naturally come from the approach of the ions to a saturated state of orientation.

11. The impressed electric field does its work, under ordinary conditions, by turning the ions so that the pits upon them, wherever they at first occur, move as a positive charge would move, in the conventional direction of the field force, while the electron portion of the damaged surface shell of the ion moves as a whole in the opposite direction. Thus we have an electric *current*. The effect of the to and fro heat translations of the ion, superposed on the process of orientation, seems to be, practically, to enlarge the ion.

12. The rapidity of the process of orientation of an ion is probably controlled by various factors, one of which is, of course, the moment of inertia of the ion, but the time occupied in this process is assumed to be less than the normal life time of an ion. See (21).

Ohm's Law

13. Let the condition of the ion i in figure 1 represent the completed orientation due to a field the lines of which run in the direction of F. If the atom a, now in contact with i, loses an electron to i, a pit will form

on a, and the transfer will have substituted for an ion of negative potential energy an ion of equal but opposite orientation, that is, an ion of positive potential energy of orientation, and the new ion will at once begin to swing in such a way as to resume the process that had come to an end on

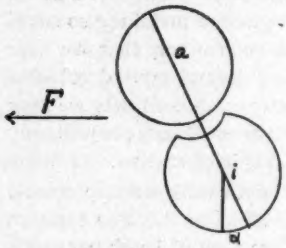

FIGURE 1

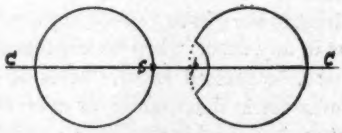

FIGURE 2

the first ion. Other things being equal, the strength of the current will be proportional to the sine of the angle of orientation, α in figure 1, which through a wide range of field strength is (see Prop. (5)) practically proportional to this strength. Thus Ohm's law is satisfied, so far as the kind of conduction now under consideration, that is, the "associated-electron conductivity," which in my previous papers I have called k_a. The "free electron conductivity," k_f, is to be considered briefly later in this paper.

The Temperature Relation

14. It is misleading to speak of the conductivity of pure metals as being proportional to the reciprocal of the absolute temperature. If this relation held, the temperature coefficient of resistance would be 0.00367 for all metals, whereas its mean value between 0° C. and 100° C. is greater than 0.00367 in most metals and is greater than 0.006 in some. Moreover, the purer the metal, the higher the value of this coefficient. Accordingly a theory of conduction that makes conductivity inversely proportional to T proves too much. It is probable that the observed rather loose relationship between conductivity and temperature is the outcome of opposing influences, the direct effect of rise of temperature being to increase resistance much more rapidly than the temperature increases, while increase in the number of ions with heating of the metal tends to reduce the resistance. This idea is elaborated in the following paragraphs.

15. The necessary relative position of an ion and an atom, with a view to the passage of an electron from the atom to the ion, is indicated by figure 2, where the circle represents an atom, the imperfect circle an ion, p the point at which an electron must lodge in order to make an atom of the

ion, *cc* the line of centres, and *s* a little three dimensional space within which the point *p* must be in order to capture an electron from the atom. It is to be noted that the pit in *i* allows the point *p* to pass *through s* during a very close approach of the ion and the atom.[6]

16. Let us for convenience suppose the atom to remain at rest, all the relative motion of the two particles being, for the present discussion, attributed to the ion. Ignoring any small long-period rotary oscillations the ion may have, we see that its vibrations may be resolved along three axes, one the line *cc*, and the other two at right angles with *cc* and with each other. The little vital space *s*, within which the point *p* of the ion must be in order to take up an electron from the atom, lies at the middle of the second and third of these component vibrations but nearer one end than the other of the vibration parallel to *cc*.

17. The probability that *p* will gain an electron during any single passage through *s* is proportional to the length of time spent in this passage. It is, then, so far as the second and third axes are concerned, inversely proportional to the product of the amplitudes of vibration parallel to these axes, each of which amplitudes is proportional to $T^{0.5}$. As to the vibrations parallel to *cc*, the space *s* lies relatively nearer the end in a short vibration than in a long one. Accordingly it may be assumed, provisionally, that, so far as motion along *cc* is concerned, the time spent by *p* in passing through *s* will be inversely proportional to some power of *T* higher than $T^{0.5}$. The result, all three component vibrations being considered, is a probability, that *p* will gain an electron in a single passage through *s*, proportional to T^{-x}, where *x* is some quantity a little greater than 1.5.

18. The conductivity under discussion should, other things being equal, be proportional to the amount of orientation suffered by the ions, which according to (5) varies as T^{-1} and to the frequency of transmission contacts between ions and atoms, which according to (17) varies as T^{-x}. Hence if *n*, the number of ions per cu. cm., remained constant, we should have this conductivity proportional to $T^{-(1+x)}$ where $(1+x)$ is somewhat greater than 2.5.

19. In the series of papers on thermo-electric action, thermal conduction, etc., which I have published in these PROCEEDINGS[7] during the past few years, I have been led to the conclusion that the value of *n* can be usefully represented by the formula $n = zT^q$, where *z* and *q* are constants, the value of the latter varying in the many metals considered, from 1.2 in bismuth and in iron to 1.6 in nickel, palladium and platinum, the mean being 1.48. This general rate of increase of *n* with rise of temperature, taken with the propositions of (18), suggests that the conductivity k_a, now under discussion, should vary, approximately, inversely as the absolute temperature, though individual metals might depart rather widely from this relation. The relation in question is:

$$k_a \alpha T^{-(1+x-q)}$$

20. Evidently it is possible, by taking for each metal an arbitrary value of x, to make the ratio of k_a at 0° C. to k_a at 100° C. agree with the observed ratio of total conductivity at 0° C. to total conductivity at 100° C., but this would be a rather fantastic performance. Attempts to account exactly, at present, by means of the theory of conduction presented in this paper, for the details of observed fact in the relationship of conductivity to temperature would be the opposite of convincing. The value 1.6 for x serves very well for many metals.

21. I cannot in this paper reproduce the discussion by which I have sought to test the quantitative adequacy of the theory here presented. This discussion involves necessarily considerable additional conjecture and is not yet to be regarded as conclusive, but it is distinctly encouraging, provided it is allowable to use for certain factors values of an order of magnitude indicated below:

Moment of inertia of an atom, 10^{-42} gm., cm.,[2]

Time of orientation process, 5000 times the period of "characteristic" heat vibrations.

Life-time of an ion, at 0° C., twice the orientation time.

n, the number of free electrons or of ions, per cu. cm., in general accord with the data given in my *Summary,*[7] though I am now inclined to take, at 0° C., the mean path of a "free electron" within a metal as perhaps fifty times the centre to centre distance of the atoms, instead of ten times, and so get a correspondingly smaller number for n, one of the quantities varying inversely as the other.

Under these conditions the mean kinetic energy of rotary motion of an atom (see (5)) comes out about 10^{-9} times the energy of translatory vibration.

Behavior of the Free Electrons in Conduction

22. According to the data given in any *Summary,* free-electron conductivity bears to total conductivity about the same ratio at 100° C. as at 0° C., that is, in most metals the free-electron conductivity, k_f, is roughly inversely proportional to the absolute temperature.

23. Various writers give us the formula

$$k_f = \frac{n \lambda c e^2}{\kappa R T},$$

though they may differ in the value assigned to the constant κ. If we take as approximately true, the proposition that $k_f T$ is a constant, we have

$n\lambda c$ equally constant, and so, if we take $n \propto T^{1.5}$ (see (19)), we have $\lambda \propto T^{-2}$, approximately.

24. In the attempt, not hopeless, which I have been making of late to explain the various "transverse" effects in accordance with the propositions of my *Summary*, I have been led to the conclusion that the diameter of the outer electron shell of an atom may be as small as one-half the means centre to centre distance of neighboring atoms. This conception makes it comparatively easy to accept the proposition that λ may be, as already indicated in (21), fifty[8] times as great as this centre to centre distance.

If λ really extends through many one-atom-thick layers of the metal, a slight decrease in the probability of an electron's passing through any single layer would be enough to account for the rapid decrease in λ with use of temperature, as found in (23).

[1] "A Possible Function of the Ions in the Electric Conduction of Metals," PROC. NAT. ACAD. SCI., Vol. 3, March 1917.

[2] *Journal de Physique*, series 4, vol. 4, pp. 690–692.

[3] See pp. 86–88 of his *Molecular Constitution of Matter*.

[4] Borelius makes a like assumption regarding the kinetic energy of what he calls free electrons. See *Ann. Phys. Chem.*, vol. 57, p. 233.

[5] In the paper referred to in footnote (1), I held quite the opposite opinion, making the effect of the field at this juncture the basis of my explanation of Ohm's law.

[6] If this conception is sound, it would not be surprising to find very great pressure unfavorable to the state of superconductivity.

[7] For a *Summary* see PROC. NAT. ACAD. SCI., 7, No. 3, March 1921.

[8] Bridgman has of late taken λ to be "many" times the atomic diameter.

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*STATIC DEFLECTION, LOGARITHMIC DECREMENT AND FIRST SEMI-PERIOD OF THE VACUUM GRAVITATION NEEDLE**

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1. *Apparatus.*—The object of the present experiments is a completion of the work of an earlier paper,[1] by carrying the exhaustion of the case as far as practicable.

In spite of the care taken to seal all parts of the extended apparatus, there remained a constant leak of .0035 mm. of mercury per hour, the seat of which I was unable to detect; but as experiments with gradually decreasing vacua were primarily contemplated, this leak was here no serious

disadvantage, except that the highest exhaustions were not available. It is of course possible to keep the pump running and the stopcock open. When this is done, however, the needle (quite apart from tremor) is always in motion, so that gravitation measurements are out of the question.

2. *Data. Logarithmic Decrement,* λ.—In the work of last year the logarithmic decrement as obtained from observations largely made at night, seemed to remain fairly constant until the highest exhaustions were approached. This result needed the further qualification undertaken in the present paper. To compute $\lambda \log e$, the two arcs obtained from the first three elongations of the needle $y - y'$, $y' - y''$ were used. In view of the specified small leak in the apparatus, the data for λ could be found with great accuracy in this way in the course of time, while the vacuum slowly degenerated from about .0002 to .271.

These data and the vacua to which they belong (observations being made in the morning (A), afternoon (P), and night (N)), are constructed in relation to the vacua (mm. of mercury) in figure 1. F denotes that the attracting weight M, on the right, is in front, and R to the rear of the case and needle, the opposite conditions holding for the M near the left end of the needle. The first arc of swing was usually between $y = 20$ and $y = 30$ cm., depending on the time of day.

Inspection of figure 1 shows that changes of viscosity due to changes of pressure, are not discernible in results of the present kind. In fact the logarithmic decrement at the highest exhaustions (July 5) happens to be larger than at the lowest exhaustions (July 8, N). The logarithmic decrement is therefore, also, primarily under the influence of the radiant forces. It is largest in the morning, and least at night. It obviously oscillates once per day, though I did not make observations after 11 P.M. If $d\Theta/dt$ be the change of atmospheric temperature per second in the environment of the apparatus, then this coefficient is the controlling factor in the marked variations of $\lambda \log e$.

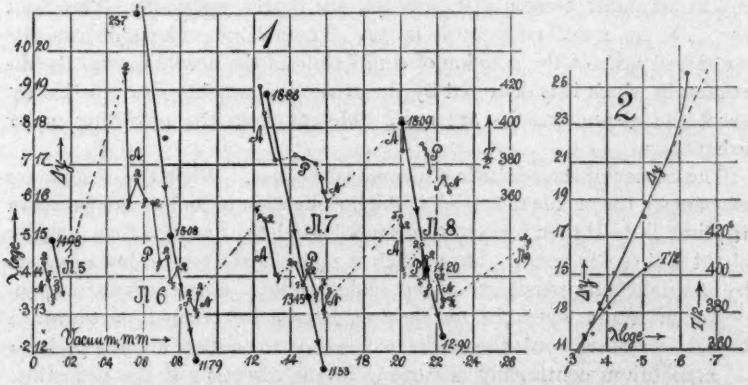
Moreover the values of $\lambda \log e$ are always larger for the rear positions R of M, than for the front positions F, under otherwise like conditions. One or the other, or their mean, must therefore be taken in association with $d\Theta/dt$. This may be due to a lack of symmetry of the position of the needle to the case; but it is more probably due to a lack of symmetry in the environment.

3. *First Semi-Periods.*—The accurate method of finding $T/2$ from two successive passages of the needle through the position of equilibrium is very troublesome; for $T/2$ is long and the position of equilibrium varies. Similarly the variations of $T/2$ are large, and precise values of it are here apparently of little use. Hence the measurement of $T/2$ from elongation to elongation was accepted as adequate. These values, so far as taken,

are constructed in figure 1, on the right, in seconds. The variations may run from 6 to 7 minutes and the $T/2$ values are largest in the morning and least at night. They march about in parallel to the λ values, if either the R or the F position alone is taken. T thus also varies harmonically in the lapse of time with a period of 24 hours referable to $d\Theta/dt$, provided the first semi-period, $T/2$ is taken.

From these results it might be surmised that the true period T can be computed from the log decrement observed at the same time. But it is easily seen that the correction so obtained is of an order entirely too small to account for the existing divergencies. The theory of the damped pendulum is only useful here as an analogy.

4. *Static Displacements*, Δy.—Continuous series of these data were given in the earlier papers and obtained from night observations (7–11 P.M.), when they are liable to be most constant. The black dots in figure 1 are the present values for different parts of the same day. Δy is the



static displacement finally obtained, when the attracting weights M are swung from one side of the needle to the other. As figure 1 shows, Δy is now also markedly harmonic in the lapse of time, or of the reduced exhaustion proportional to time, with a period of one day.

We thus come to the result that λ, $T/2$, Δy are similar time functions; that as a first approximation they are independent of the change of viscosity of rarified air and that their relative variation is such as to admit of their expression in terms of each other. It does not necessarily follow that a correct value computed for one (for instance for λ) would lead to correct values for $T/2$ and Δy; but it is a project well worth testing.

5. *Comparison of* λ, $T/2$, Δy.—It is next necessary, therefore, to more specifically compare all the quantities obtained in the present investigation. This has been done in figure 2, where the abscissas are $\lambda \log e$ and the ordinates $T/2$ and Δy, respectively.

Turning first to the latter, it is astonishing to find that λ and Δy, in spite of the marked amplitude of radiant forces, make up a definite graph. In fact, if we omit the exceptional $\Delta y = 25.7$ (inadmissably large radiant forces), Δy and λ are nearly proportional. If this curve were known, Δy could be computed from λ and vice versa.

The values of $T/2$, as already intimated, were not accurately taken, as I did not anticipate relations like the present. Nevertheless figure 2 adequately shows that these also make out a definite graph.

If for instance $\Delta y = 13.4$ cm.(about the mean value in the preceding paper), $T_1 = 758$ sec. should be its equivalent and $\lambda \log e = .385$ at the same time. If we take the vacuum period as $T_1/\sqrt{1+\lambda^2/4\pi^2}$ it would be $T = 750$ sec. which happens to agree with the value taken in the report in question.

6. *Plenum.*—In contrast with the preceding oscillations under conditions of high exhaustion, the results investigated for the case of the needle vibrating in air under atmospheric pressure, are nearly aperiodic. The data for T, λ, Δy, are all enormously larger. These values of λ result from the continued drift of the position of equilibrium of the needle toward the direction in which it is deflected by the attracting weights M. In this respect the plenum results are quite different from the preceding under exhaustion.

The motion of the needle in a plenum is peculiar. When the weights are exchanged, the needle falls from its high elongation to its low elongation in the time $T/2$. It then turns toward a new high elongation for a time (usually about $T/4$ sec. in length), after which it again moves toward low numbers by slow indefinite creeping, without again turning. Exactly the same phenomenon occurs when the needle rises from a low elongation toward a higher. In other words the needle may be said to oscillate about a position of equilibrium continually advancing in the direction of the deflection.

7. *Comparison with Theoretical Values of Frictional Resistance.*—It is now desirable to endeavor to construe the values obtained for λ and T by the aid of the familiar theory of the damped pendulum. If the needle be regarded as consisting of the two balls at its end, the frictional forces should be $6\pi\eta r v$ and therefore the frictional coefficient b is $6\pi\eta r$. If T_1 is the period of the damped needle and λ its logarithmic decrement $T_1 b = 2\lambda$, and therefore, $\lambda = 3\pi\eta r T_1$. Hence, if $r = .23$ cm., $\eta = .00019$, $T_1 = 892\,\lambda \log e$ seconds. In figure 2, however, not only is proportionality of T and λ at long range excluded, but the factor would be twice as large, on the average.

* Advance note, from a Report to the Carnegie Institution of Washington, D. C.

[1] These PROCEEDINGS, 8, pp. 13, 63, 1922. The present apparatus is essentially the same.